

A NOTE ON THICK SUBCATEGORIES OF STABLE DERIVED CATEGORIES

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ABSTRACT. For an exact category having enough projective objects, we establish a bijection between thick subcategories containing the projective objects and thick subcategories of the stable derived category. Using this bijection we classify thick subcategories of finitely generated modules over local complete intersections and produce generators for the category of coherent sheaves on a separated noetherian scheme with an ample family.

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1. INTRODUCTION

Let A be a not necessarily commutative ring and $\mathbf{mod} A$ be the category of finitely presented A -modules. Consider the *stable derived category* of A in the sense of Buchweitz [6], which is also called the *triangulated category of singularities* or just the *singularity category*, following work of Orlov [13]. This category is by definition the Verdier quotient of the bounded derived category of $\mathbf{mod} A$ with respect to the triangulated subcategory consisting of all perfect complexes:

$$D^b(\mathbf{mod} A)/D^b(\mathbf{proj} A).$$

In a number of recent papers, thick subcategories of this triangulated category have been studied and even classified, typically in terms of primes ideals of some appropriate cohomology ring [3, 4, 12, 16, 17]. In this note we point out a bijection between thick subcategories of the stable derived category and thick subcategories of the module category containing all projective modules. In a special case this bijection has been observed by Takahashi [18]. Using our more general form of this bijection we are able to extend Takahashi's Theorem 4.6 (1) to complete intersections.

Further applications of this bijection arise from the study of generators of exact and triangulated categories. We illustrate this by results of Oppermann–Stovicek [12] and Schoutens [15]. In the latter case, we include a generalisation as well as a new proof.

2. THICK SUBCATEGORIES VERSUS THICK SUBCATEGORIES

Let \mathcal{A} be an exact category in the sense of Quillen [14] and denote by $D^b(\mathcal{A})$ its bounded derived category [9, 19]. Suppose that \mathcal{A} has enough projective objects. Let $\text{Proj } \mathcal{A}$ be the full subcategory consisting of the projective objects and view $D^b(\text{Proj } \mathcal{A})$ as a thick subcategory of $D^b(\mathcal{A})$. The Verdier quotient

$$D^b(\mathcal{A})/D^b(\text{Proj } \mathcal{A})$$

is by definition the *stable derived category* of \mathcal{A} .

We are interested in thick subcategories of the stable derived category and observe that they correspond bijectively to thick subcategories of $D^b(\mathcal{A})$ containing all projective objects. Recall that a full additive subcategory of a triangulated category is *thick* if it is closed under shifts, mapping cones, and direct summands.

A full additive subcategory \mathcal{C} of \mathcal{A} is called *thick* if it is closed under direct summands and has the following two out of three property: for every exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{A} with two terms in \mathcal{C} , the third term belongs to \mathcal{C} as well.

In the following we identify \mathcal{A} with the full subcategory of $D^b(\mathcal{A})$ consisting of all complexes concentrated in degree zero.

Theorem 1. *Let \mathcal{A} be an exact category having enough projective objects. The map sending a subcategory \mathcal{D} of $D^b(\mathcal{A})$ to $\mathcal{D} \cap \mathcal{A}$ induces a bijection between*

- *the thick subcategories of $D^b(\mathcal{A})$ containing all projective objects, and*
- *the thick subcategories of \mathcal{A} containing all projective objects.*

The inverse map sends a thick subcategory \mathcal{C} of \mathcal{A} to $D^b(\mathcal{C})$.

Proof. We may assume that \mathcal{A} is idempotent complete, keeping in mind that the idempotent completion of $D^b(\mathcal{A})$ equals the bounded derived category of the idempotent completion of \mathcal{A} ; see [1].

The first part of this proof is taken from the appendix of [2]. Suppose that \mathcal{C} is a thick subcategory of \mathcal{A} containing all projectives. The category \mathcal{C} inherits an exact structure from \mathcal{A} and the inclusion $\mathcal{C} \rightarrow \mathcal{A}$ induces therefore an exact functor $D^b(\mathcal{C}) \rightarrow D^b(\mathcal{A})$. The fact that \mathcal{C} contains the projective objects implies that this functor is fully faithful; see for instance [19, Proposition III.2.4.1]. Thus the full subcategory \mathcal{D} of $D^b(\mathcal{A})$ consisting of objects isomorphic to a complex of objects in \mathcal{C} is a thick subcategory. We claim that $\mathcal{C} = \mathcal{D} \cap \mathcal{A}$. Clearly, $\mathcal{C} \subseteq \mathcal{D} \cap \mathcal{A}$. Thus we fix $X \in \mathcal{D} \cap \mathcal{A}$. Then X is in $D^b(\mathcal{A})$ isomorphic to a bounded complex C with differential δ such that $C^n \in \mathcal{C}$ for all n and C is acyclic in all degrees $n \neq 0$. Now we use that \mathcal{C} is thick. Thus $\text{Coker } \delta^{-2}$ and $\text{Ker } \delta^0$ belong to \mathcal{C} , and we have an admissible monomorphism $\text{Coker } \delta^{-2} \rightarrow \text{Ker } \delta^0$ such that the cokernel is isomorphic to X . We conclude that X belongs to \mathcal{C} , and therefore $\mathcal{C} = \mathcal{D} \cap \mathcal{A}$.

Now fix a thick subcategory \mathcal{D} of $D^b(\mathcal{A})$ containing all projectives and set $\mathcal{C} = \mathcal{D} \cap \mathcal{A}$. Each exact sequence $0 \rightarrow X \rightarrow Y \rightarrow Z \rightarrow 0$ in \mathcal{A} gives rise to an exact triangle $X \rightarrow Y \rightarrow Z \rightarrow X[1]$ in $D^b(\mathcal{A})$. Thus \mathcal{C} is a thick subcategory of \mathcal{A} . We claim that the functor $D^b(\mathcal{C}) \rightarrow \mathcal{D}$ is an equivalence. In view of the first part of the proof, it suffices to show that each object X in \mathcal{D} is isomorphic to a complex of objects in \mathcal{C} . We may assume that X is a complex of projective objects with $X^n = 0$ for $n \gg 0$ and X acyclic in degrees $n \leq p$ for some integer p . Truncating X in degree p gives an exact triangle $X' \rightarrow X \rightarrow X'' \rightarrow X'[1]$ such that X' is a bounded complex of projective objects and X'' is acyclic in all degrees different from p . Clearly, X' belongs to $D^b(\mathcal{C})$, and X'' is isomorphic to a shift of an object from \mathcal{C} . It follows that X is isomorphic to an object in $D^b(\mathcal{C})$. \square

Remark 2. As noted earlier thick subcategories of the stable derived category correspond to thick subcategories of $D^b(A)$ containing the projective objects. It follows that the theorem gives a bijection between thick subcategories of the stable derived category and thick subcategories of A containing all projective objects.

Remark 3. In the theorem and its proof, one can replace the subcategory $\text{Proj } A$ by any thick subcategory P of A having the property that each object $X \in A$ admits an admissible epimorphism $P \rightarrow X$ with $P \in P$.

Remark 4. A full additive subcategory C of A is thick and contains all projective objects iff it is closed under taking extensions, direct summands, cosyzygies and syzygies.

3. APPLICATIONS AND COMMENTS

In this section we make some brief remarks concerning applications of the bijection of the last section. In particular, we extend the results of Schoutens and Takahashi as promised in the introduction.

3.1. Classification theorems. We will explain how to use the main theorem to obtain two classification results, one new and one known, for thick subcategories of abelian categories.

We first use Theorem 1 to give a classification of the thick subcategories of $\text{mod } A$, which contain A , when A is a local complete intersection ring; this extends [18, Theorem 4.6 (1)]. In order to state the classification we need to introduce a hypersurface associated to A .

Let A be a local *complete intersection* i.e., there is a regular local ring B and a surjection $B \rightarrow A$ with kernel generated by a regular sequence. Set $\mathbb{X} = \text{Spec } A$, $\mathbb{T} = \text{Spec } B$, $\mathcal{E} = \mathcal{O}_{\mathbb{T}}^{\oplus c}$, and $t = (b_1, \dots, b_c)$ where the b_i form a regular sequence generating the kernel of $B \rightarrow A$. Let \mathbb{Y} be the hypersurface defined by the section $\Sigma_{i=1}^c b_i x_i$ of $\mathcal{O}_{\mathbb{P}_B^{c-1}}(1)$ where the x_i form a basis for the free B -module $H^0(\mathbb{P}_B^{c-1}, \mathcal{O}_{\mathbb{P}_B^{c-1}}(1))$. In summary we are concerned with the following commutative diagram

$$\begin{array}{ccc} \mathbb{P}_A^{c-1} & \xrightarrow{i} & \mathbb{Y} \xrightarrow{u} \mathbb{P}_B^{c-1} \\ p \downarrow & & \downarrow q \\ \mathbb{X} & \xrightarrow{j} & \mathbb{T}. \end{array}$$

The following corollary of Theorem 1, based upon [16, Corollary 10.5], shows that the hypersurface \mathbb{Y} controls the thick subcategories of $\text{mod } A$ which contain the projectives.

Corollary 5. *Let A be a local complete intersection and let \mathbb{Y} be the hypersurface as defined above. Then there is an order preserving bijection*

$$\left\{ \begin{array}{l} \text{specialisation closed} \\ \text{subsets of } \text{Sing } \mathbb{Y} \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{l} \text{thick subcategories of} \\ \text{mod } A \text{ containing } A \end{array} \right\},$$

where $\text{Sing } \mathbb{Y}$ denotes the set of $y \in \mathbb{Y}$ such that the local ring $\mathcal{O}_{\mathbb{Y}, y}$ is not regular.

Proof. By Theorem 1 there is a bijection between thick subcategories of $\text{mod } A$ containing A and thick subcategories of $D^b(\text{mod } A)$ containing A . As stated in Remark 2 the thick subcategories of $D^b(\text{mod } A)$ containing A are in bijection with the thick subcategories of the stable derived category $D^b(\text{mod } A)/D^b(\text{proj } A)$. The result now follows from the classification of thick subcategories of the stable derived category given in [16, Corollary 10.5]. \square

In a similar vein we can apply the theorem to the work of Benson, Carlson, and Rickard [3] on stable categories in modular representation theory. Let G be a finite group and let k be a field whose characteristic divides the order of G . We denote the category of finite dimensional representations of kG by $\mathbf{mod} kG$ and by $\mathbf{stmod} kG$ its stable category which is obtained by annihilating all projective modules. We say a subcategory \mathcal{C} of $\mathbf{mod} kG$ is a *thick tensor ideal* if it is a thick subcategory which is closed under tensoring with arbitrary objects of $\mathbf{mod} kG$. Applying Theorem 1 to [3, Theorem 3.4] immediately recovers the following known result.

Theorem 6. *Let k and G be as above. Then there is an order preserving bijection*

$$\left\{ \begin{array}{c} \text{specialisation closed} \\ \text{subsets of } \mathrm{Proj} H^*(G, k) \end{array} \right\} \begin{array}{c} \longrightarrow \\ \longleftarrow \end{array} \left\{ \begin{array}{c} \text{thick tensor ideals of} \\ \mathbf{mod} kG \text{ containing } kG \end{array} \right\}$$

where $H^*(G, k)$ denotes the cohomology of G with coefficients in k .

Although this is implicit in the work of Benson, Carlson, and Rickard it does not seem to be explicitly stated in the literature. One should also compare this with the analogue for the category of all kG -modules which appears as Theorem 10.4 in [4].

3.2. Generating categories of coherent sheaves on schemes. The *singular locus* $\mathrm{Sing} A$ of a commutative noetherian ring A consists of all prime ideals \mathfrak{p} such that the localisation $A_{\mathfrak{p}}$ is not regular. More generally the *singular locus* $\mathrm{Sing} \mathbb{X}$ of a scheme \mathbb{X} consists of those points $x \in \mathbb{X}$ such that the local ring $\mathcal{O}_{\mathbb{X},x}$ is not regular. We give a new proof of the following result.

Theorem 7 (Schoutens [15]). *Let A be a commutative noetherian ring. Then the smallest thick subcategory of the category of A -modules containing A and A/\mathfrak{p} for all \mathfrak{p} in the singular locus of A equals the category of noetherian A -modules.*

As a corollary we will prove a generalisation of this result for schemes having an ample family of line bundles. Keeping this generalisation in mind, let us fix a separated noetherian scheme \mathbb{X} and let $\{\mathcal{L}_i \mid 1 \leq i \leq n\}$ be an ample family of line bundles on \mathbb{X} . Recall that a family of line bundles $\{\mathcal{L}_i \mid 1 \leq i \leq n\}$ on \mathbb{X} is *ample* if there is a family of sections $f \in H^0(\mathbb{X}, \mathcal{L}_i^{\otimes m})$ with $1 \leq i \leq n$ and $m > 0$ such that the $\mathbb{X}_f = \{x \in \mathbb{X} \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x^{\otimes m}\}$, where \mathfrak{m}_x is the maximal ideal of $\mathcal{O}_{\mathbb{X},x}$, form an open affine cover of \mathbb{X} . Of course one can just keep in mind the example of an affine scheme $\mathbb{X} = \mathrm{Spec} A$ and take $\{\mathcal{O}_{\mathbb{X}}\}$ for the ample family. We denote by $\mathrm{coh} \mathbb{X}$ the abelian category of coherent sheaves of $\mathcal{O}_{\mathbb{X}}$ -modules.

The proof passes through the homotopy category of injective sheaves of $\mathcal{O}_{\mathbb{X}}$ -modules. Let $\mathbf{K}(\mathbb{X})$ denote the homotopy category of complexes of quasi-coherent $\mathcal{O}_{\mathbb{X}}$ -modules and $\mathbf{K}(\mathrm{Inj} \mathbb{X})$ the full subcategory consisting of complexes of injective quasi-coherent $\mathcal{O}_{\mathbb{X}}$ -modules. We identify each quasi-coherent sheaf with the corresponding complex in $\mathbf{K}(\mathbb{X})$ concentrated in degree zero.

Lemma 8. *Let \mathbb{X} be a noetherian scheme as above and \mathcal{C} a subcategory of $\mathrm{coh} \mathbb{X}$ containing the sheaves $\{\mathcal{L}_i^{\otimes m} \mid 1 \leq i \leq n, m \in \mathbb{Z}\}$. Suppose that any complex Y of injective quasi-coherent $\mathcal{O}_{\mathbb{X}}$ -modules is nullhomotopic provided that*

$$\mathrm{Hom}_{\mathbf{K}(\mathbb{X})}(X, Y[n]) = 0 \quad \text{for all } X \in \mathcal{C}, n \in \mathbb{Z}.$$

Then the smallest thick subcategory of coherent sheaves containing \mathcal{C} is $\mathrm{coh} \mathbb{X}$.

Proof. The functor $\mathbf{i}: \mathrm{coh} \mathbb{X} \rightarrow \mathbf{K}(\mathrm{Inj} \mathbb{X})$ taking a sheaf to its injective resolution extends to an equivalence $\mathrm{D}^b(\mathrm{coh} \mathcal{O}_{\mathbb{X}}) \xrightarrow{\sim} \mathbf{K}(\mathrm{Inj} \mathbb{X})^c$ onto the full subcategory of

compact objects of $\mathbf{K}(\mathrm{Inj} \mathbb{X})$, by [8, Proposition 2.3]. Note that

$$\mathrm{Hom}_{\mathbf{K}(\mathbb{X})}(\mathbf{i}X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathbf{K}(\mathbb{X})}(X, Y) \quad \text{for all } Y \in \mathbf{K}(\mathrm{Inj} \mathbb{X}),$$

by [8, Lemma 2.1]. The assumption on \mathbf{C} implies that the thick subcategory of $\mathbf{K}(\mathrm{Inj} \mathbb{X})$ generated by $\mathbf{i}(\mathbf{C})$ equals $\mathbf{K}(\mathrm{Inj} \mathbb{X})^c$. This follows from a standard argument involving Bousfield localisation; see [10, Lemma 2.2]. Thus the correspondence in Theorem 1, together with Remark 3, implies that \mathbf{C} generates $\mathrm{coh} \mathbb{X}$. \square

Proof of Theorem 7. In view of Lemma 8, it suffices to show that each complex X of injective A -modules is nullhomotopic provided that

$$\mathrm{Hom}_{\mathbf{K}(A)}(A, X[n]) = 0 \quad \text{and} \quad \mathrm{Hom}_{\mathbf{K}(A)}(A/\mathfrak{p}, X[n]) = 0$$

for all $n \in \mathbb{Z}$ and $\mathfrak{p} \in \mathrm{Sing} A$.

Thus we fix a complex X satisfying these conditions. The first one implies that X is acyclic. We may assume that X is homotopically minimal, that is, there is no non-zero direct summand of X which is nullhomotopic; see [8, Appendix B]. This means that for each $n \in \mathbb{Z}$ the truncated complex

$$X^n \rightarrow X^{n+1} \rightarrow X^{n+2} \rightarrow \dots$$

yields a minimal injective resolution of $Z^n(X)$. Localising at a prime ideal \mathfrak{p} preserves this property. If $\mathfrak{p} \notin \mathrm{Sing} A$, then $Z^n(X)_{\mathfrak{p}}$ is injective, and therefore $Z^{n+1}(X)_{\mathfrak{p}} = 0$. Thus $X_{\mathfrak{p}} = 0$. Now consider for each $n \in \mathbb{Z}$ the exact sequence

$$0 \rightarrow Z^n(X) \rightarrow X^n \rightarrow Z^{n+1}(X) \rightarrow 0$$

of modules supported on $\mathrm{Sing} A$. If this sequence does not split, one finds a finitely generated submodule U of $Z^{n+1}(X)$ isomorphic to A/\mathfrak{p} for some $\mathfrak{p} \in \mathrm{Sing} A$ such that $\mathrm{Ext}_A^1(U, Z^n(X)) \neq 0$. This follows from Baer's criterion, and one uses that the prime ideals in $\mathrm{Sing} A$ form a specialisation closed subset of $\mathrm{Spec} A$. Now observe that

$$\mathrm{Hom}_{\mathbf{K}(A)}(-, X[n+1]) \cong \mathrm{Ext}_A^1(-, Z^n(X)).$$

Thus the assumption on X implies that it is nullhomotopic. \square

Corollary 9. *Let \mathbb{X} be a separated noetherian scheme and let $\{\mathcal{L}_i \mid 1 \leq i \leq n\}$ be an ample family of line bundles on \mathbb{X} . Then the smallest thick subcategory of $\mathrm{coh} \mathbb{X}$ containing the set of coherent $\mathcal{O}_{\mathbb{X}}$ -modules*

$$\mathcal{S} = \{\mathcal{L}_i^{\otimes m} \otimes_{\mathcal{O}_{\mathbb{X}}} \mathcal{O}_{\mathcal{V}(x)}^{\alpha} \mid 1 \leq i \leq n, m \in \mathbb{Z}, x \in \mathrm{Sing} \mathbb{X}, \alpha \in \{0, 1\}\}$$

where $\mathcal{O}_{\mathcal{V}(x)}$ denotes the structure sheaf of the closed subset $\mathcal{V}(x)$ endowed with the reduced induced scheme structure and $\mathcal{O}_{\mathcal{V}(x)}^0 = \mathcal{O}_{\mathbb{X}}$, is the whole category of coherent $\mathcal{O}_{\mathbb{X}}$ -modules.

Proof. We proceed essentially as in the theorem i.e., we show that if X is a complex in $\mathbf{K}(\mathrm{Inj} \mathbb{X})$ which is not nullhomotopic then some object of \mathcal{S} maps to a suspension of X . So let us fix such a complex X . We may assume that X is acyclic as the tensor powers of the sheaves in the ample family generate the derived category (see [11, Example 1.11]) and thus detect any complex having non-zero cohomology.

Since X is not nullhomotopic there exists a $j \in \mathbb{Z}$ such that $Z^j(X)$ is not injective. Let $f \in H^0(\mathbb{X}, \mathcal{L}_i^{\otimes m})$ be a section such that $\mathbb{X}_f = \{x \in \mathbb{X} \mid f_x \notin \mathfrak{m}_x \mathcal{L}_x^{\otimes m}\}$ is an open affine on which $Z^j(X)|_{\mathbb{X}_f}$ is not injective; we can find such an f by ampleness of the family of line bundles. In particular, $X|_{\mathbb{X}_f}$ is non-zero in $\mathbf{K}(\mathrm{Inj} \mathbb{X}_f)$. Hence, by the proof of the theorem, there is an $x \in \mathrm{Sing} \mathbb{X}$ and a non-zero map

$g \in \operatorname{Hom}_{\mathbf{K}(\mathbb{X}_f)}(\mathcal{O}_{\mathcal{V}(x)}|_{\mathbb{X}_f}, X|_{\mathbb{X}_f}[j+1])$ for some integer j . By [7, Lemma III.5.14] there exists an $m' \geq 0$ so that $f^{m'}g$ lifts to a morphism

$$\tilde{g}: \mathcal{O}_{\mathcal{V}(x)} \rightarrow X \otimes_{\mathcal{O}_X} \mathcal{L}_i^{\otimes m'}[j+1].$$

The map \tilde{g} is not nullhomotopic as if it were this would yield a nullhomotopy for $f^{m'}g$ and thus g . It just remains to note that, by adjunction, the map \tilde{g} is equivalent to a non-zero morphism $\mathcal{O}_{\mathcal{V}(x)} \otimes_{\mathcal{O}_X} \mathcal{L}_i^{\otimes -m'} \rightarrow X[j+1]$ witnessing the fact that X is not nullhomotopic. \square

3.3. Strong generators. Strong generators of triangulated categories were introduced by Bondal and Van den Bergh [5, §2.2]. There seems to be no obvious analogue for exact categories. So it would be interesting to translate the following result into a statement about abelian categories, using the bijection from Theorem 1.

Theorem 10 (Opfermann–Stovicek [12]). *Let k be a field and \mathbf{A} a k -linear abelian category which is Hom-finite and has enough projective objects. If \mathbf{D} is a thick subcategory of $\mathbf{D}^b(\mathbf{A})$ which contains all projective objects and admits a strong generator, then $\mathbf{D} = \mathbf{D}^b(\mathbf{A})$.* \square

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